Dynamic Epistemic Logic for Budget-Constrained Agents

Vitaliy Dolgorukov[1] and Maksim Gladyshev[2(✉)]

[1] HSE University, Myasnitskaya 20, 101000 Moscow, Russian Federation
vdolgorukov@hse.ru

[2] Utrecht University, Princetonplein 5, 3584 CC Utrecht, The Netherlands
m.gladyshev@uu.nl

Abstract. We present a static ($\mathsf{EL_{bc}}$) and dynamic ($\mathsf{DEL_{bc}}$) epistemic logic for budget-constrained agents, in which an agent can obtain some information in exchange for budget resources. $\mathsf{EL_{bc}}$ extends a standard multi-agent epistemic logic with expressions concerning agent's budgets and formulas' costs. $\mathsf{DEL_{bc}}$ extends $\mathsf{EL_{bc}}$ with dynamic modality "$[?_i A]\varphi$" which reads as "φ holds after i's question whether a propositional formula A is true". In this paper we provide a sound and complete axiomatization for $\mathsf{EL_{bc}}$ and $\mathsf{DEL_{bc}}$ and show that both logics are decidable.

Keywords: Dynamic epistemic logic · Budget-constrained agents · Knowledge representation

1 Introduction

Dynamic epistemic logic [6,12,15] is a common way of describing agents' knowledge and informational changes. But nowadays, our intuition about the nature of agents' reasoning and interaction tells us that both processes of operating with available knowledge and obtaining a new one cannot always be effortless. This natural intuition demonstrates that reasoning often becomes a resource consuming action. A lot of researchers of epistemic logic paid attention to this problem and found different approaches to formalising the idea of resource-bounded agents [10]. The wide range of existing approaches, describing non-omniscient agents, consider resources as various cognitive limits.

Non-omniscience can be described through time- and memory-constrained agents who do not necessarily know all the logical consequences of their knowledge. Some papers model such constraints through so-called inferential actions, which require agents to take explicit inference steps, spending available resources to deduce the logical consequences of their knowledge [17]. Other papers extend the idea of a bounded deliberation process with resource consuming inference actions by introducing perception [4] or rule-based models [14] and their effects on formation of agents' beliefs. The idea of resource-bounded agents, situated in agent-environment systems that takes into account agents' observations, beliefs, goals and actions, sounds promising both for philosophers and computer scientists [2]. Most contemporary papers on resource-bounded reasoning would agree that modelling of non-omniscient agents does not mean

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modelling of imperfect reasoners. On the contrary, a lot of papers argue that epistemic logic must formalise the idea that if the agent knows all necessary premises and either thinks hard enough [8] or has enough time [1,3], then they will know the conclusion. Thus, the reasoning process itself can justifiably be considered as an ongoing time-consuming [13], as well as a memory-consuming [9] process. This intuition bridges the gap between reasoning and computation process and sounds fruitful for AI research. While this is a reasonable assumption which is worth studying, both time- and memory-based approaches deal with'inner' obstacles of an agent's deliberation process. Thus, even existing papers studying resource constraints in agent-environment settings consider resources as a tool of reasoning or obtaining new information from already available agent's knowledge. At the same time, a lot of real-life scenarios demonstrate that resources can also be considered as an instrument of obtaining new, independent or already available, information from the outside. In other words, solving some tasks can require getting additional information, which is not necessarily costless. Our main goal in this paper is to consider logically omniscient reasoners who can interact with the environment (in the sense of an independent bystander) and obtain new information from this environment by spending a certain amount of resources.

A similar attempt was made by Naumov and Tao [16]. Their paper describes budget-constrained agents in epistemic settings. It catches the intuition that sometimes agents have to spend their resources to obtain the knowledge of some fact. But since their logic is static and describes resource constraints as a feature of the knowledge operator itself, this approach violates the Negative Introspection axiom, so it appears to be a S4-like system. Nevertheless, this S4-like epistemic logic is complete, with respect to S5-like structures. Our paper aims to demonstrate that reasoning about knowledge and informational change under budget constraints can be described by an S5-like system if we consider this informational change explicitly in DEL-style language.

We assume that agents can purchase information, spending some resources available to them. Intuitively, agents can ask a question "is A true?" and get a positive or negative answer. Sometimes, this question can require some resources (e.g. money). The first example that comes to mind these days obviously involves COVID-19. We can easily imagine that Agent A can be COVID-positive without knowing about it. It is also clear that Agent A can get this information by medical testing, which usually requires some amount of money, say \$20. In this situation, Agent A can buy an answer to the question 'Am I infected?' if her budget exceeds \$20. To introduce the multi-agent dimension in this example, let's assume that Agent A is a professor at some university, U. Nowadays it is common practice that professors are asked to work remotely. Imagine that our university, U can relax these restrictions and allow working on campus for those professors (agents) who can provide a negative COVID-test. It is also easy to imagine that a university can have a list of all professors who took a test (for example, this university can be in cooperation with some medical organisation). But the results of these tests are available to professors only, due to the medical privacy. Thus, if Agent A decides to take a test, she definitely obtains the result. At the same time U (1) does not know if A is infected, but it also knows that (2) 'A knows she is infected or A knows she is not infected'. But since this action requires \$20, U also knows that (3) A had at least \$20 before testing, and if U knows that A had n_1 or n_2 (where $n_1, n_2 \geq 20$), then

(4) U knows that A has \$$(n1 - 20)$ or \$$(n2 - 20)$ now. We hope that this example is clear and represents useful intuitions about resource-consuming informational updates in a multi-agent setting. Thus, we intend to model situations in which agents can spend resources in order to obtain an answer to some question. In our framework, the very fact of the question is public. i.e. every agent knows that question is asked. But the answer is private, so only one agent knows it. We also assume that resources can be understood in some abstract way, similar to the idea of utility in economics. Thus, we can consider money, effort or any other kind of agent's utility as resources in our models. We build our logic upon the standard S5 epistemic logic [12], enriched with linear inequalities described in [11] to deal with costs of the formulas and agents' budget. Then, we extend this logic with dynamic operator $[?_i A]$ combining ideas of public announcement logic [6], contingency logic with arbitrary announcement [5] and some intuitions about semi-private announcements. Section 2 of this paper deals with static epistemic logic for reasoning about costs of formulas and agent's budget. We demonstrate that this logic is sound and complete. Section 3 provides a dynamic extension of static fragment which allows us to reason about informational change for budget-constrained agents. We also state a soundness and completeness result for dynamic fragment via standard reduction argument and prove that both $\mathsf{EL_{bc}}$ and $\mathsf{DEL_{bc}}$ are decidable.

2 Epistemic Logic for Budget-Constrained Agents

Here we present the syntax and semantics of the *epistemic logic for budget-constrained agents* $\mathsf{EL_{bc}}$. In Sect. 3 we extend it with the dynamic operators for model updates.

2.1 Syntax

Let $\mathsf{Prop} = \{p, q, \dots\}$ be a countable set of propositional letters. Denote by $\mathcal{L}_{PL}$ the set of all propositional (non-modal) formulas defined by the following grammar (where p ranges over Prop, other connectives are defined standardly):

$$A, B \ ::= p \mid \neg A \mid (A \wedge B).$$

Definition 1 (The language $\mathsf{EL_{bc}}$). Let $\mathsf{Agt} = \{i, j, \dots\}$ be a finite set of agents. We fix a set of constants $\mathsf{Const} = \{c_A \mid A \in \mathcal{L}_{PL}\} \cup \{b_i \mid i \in \mathsf{Agt}\}$. It contains a constant c_A for the *cost* of each propositional formula A and a constant b_i for the *budget* of each agent i. *Formulas* of the language $\mathsf{EL_{bc}}$ are defined by the following grammar:

$$\varphi, \psi \ ::= p \mid (z_1 t_1 + \dots + z_n t_n) \geq z \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i \varphi,$$

where p ranges over Prop, $i \in \mathsf{Agt}$, $t_1, \dots, t_n \in \mathsf{Const}$ and $z_1, \dots, z_n, z \in \mathbb{Z}$.

Other Boolean connectives $\rightarrow, \vee, \leftrightarrow, \bot$ and $\top$ are defined in the standard way. The dual operator for K_i is defined as $\hat{K}_i\varphi := \neg K_i \neg\varphi$. We will also use $K_i^? \varphi$ as an abbreviation for $(K_i\varphi \vee K_i\neg\varphi)$. Note that we introduce the cost c_A only for *propositional* formulas $A \in \mathcal{L}_{PL}$. The logic with costs of arbitrary epistemic formulas is left for future research. We deal with linear inequalities and use the same abbreviations as in [11].

Thus, we write $t_1 - t_2 \geq z$ for $t_1 + (-1)t_2 \geq z$, $t_1 \geq t_2$ for $t_1 - t_2 \geq 0$, $t_1 \leq z$ for $-t_1 \geq -z$, $t_1 < z$ for $\neg(t_1 \geq z)$, and $t_1 = z$ for $(t_1 \geq z) \wedge (t_1 \leq z)$. Thus, the language $\mathsf{EL_{bc}}$ allows us to express statements such as: "$c_{p\wedge q} \geq 7$", "$b_i \geq 5$", "$2b_i = b_j$", "$K_c(b_i + b_j \geq c_{p\vee q})$" etc.

The set of *subformulas* $\mathsf{Sub}(\varphi)$ of a formula φ is defined in the standard way; note that if a constant c_A occurs in φ then we do *not* count A as a subformula of φ.

2.2 Semantics

A *model* $\mathcal{M}$ of the logic $\mathsf{EL_{bc}}$ has the components standard for the multi-modal logic $\mathbf{S5}$, namely, a non-empty set of states W, an epistemic accessibility relation $\sim_i$ for each agent $i \in \mathsf{Agt}$, and a valuation $V\colon \mathsf{Prop} \to 2^W$. Besides, a model $\mathcal{M}$ contains a function Cost that assigns to every propositional formula at each state its cost, and a function Bdg that assigns to each agent $i \in \mathsf{Agt}$ at each state $w \in W$ the available amount of resources.

Definition 2 (Kripke-style semantics). A *model* is a tuple $\mathcal{M} = (W, (\sim_i)_{i\in\mathsf{Agt}}, \mathsf{Cost}, \mathsf{Bdg}, V)$, where

- W is a non-empty set of *states*,
- $\sim_i \subseteq (W \times W)$ is an equivalence relation for each $i \in \mathsf{Agt}$,
- $\mathsf{Cost}\colon W \times \mathcal{L}_{PL} \longrightarrow \mathbb{R}^+$ is the (non-negative) *cost* of propositional formulas,
- $\mathsf{Bdg}\colon \mathsf{Agt} \times W \longrightarrow \mathbb{R}^+$ is the (non-negative) *bugdet* of each agent at each state,
- $V\colon \mathsf{Prop} \to 2^W$ is a *valuation* of propositional variables.

Thus both the cost of a formula and the budget of an agent depend on a current state. We use $\mathsf{Bdg}_i(w)$ as an abbreviation for $\mathsf{Bdg}(i, w)$, where $i \in \mathsf{Agt}$ and $w \in W$. In order to formulate additional constraints on the function Cost, we need the following notation. Let PL be the classical propositional logic. For any propositional formulas A and B:

- A and B are called *equivalent*: $A \equiv B$ iff $\vdash_{\mathrm{PL}} A \leftrightarrow B$,
- A and B are called *similar*: $A \approx B$ iff $A \equiv B$ or $A \equiv \neg B$.

We also impose the following conditions on the function Cost:

(C1) $\mathsf{Cost}(w, \bot) = \mathsf{Cost}(w, \top) = 0$,
(C2) $A \approx B$ implies $\mathsf{Cost}(w, A) = \mathsf{Cost}(w, B)$, for all $A, B \in \mathcal{L}_{PL}$ and all $w \in W$.

Definition 3. The *truth* $\vDash$ of a formula A at a state $w \in W$ of a model $\mathcal{M}$ is defined by induction:

$\mathcal{M}, w \vDash p$ iff $w \in V(p)$,
$\mathcal{M}, w \vDash \neg\varphi$ iff $\mathcal{M}, w \nvDash \varphi$,
$\mathcal{M}, w \vDash \varphi \wedge \psi$ iff $\mathcal{M}, w \vDash \varphi$ and $\mathcal{M}, w \vDash \psi$,
$\mathcal{M}, w \vDash K_i\varphi$ iff $\forall w' \in W\colon w \sim_i w' \Rightarrow \mathcal{M}, w' \vDash \varphi$,
$\mathcal{M}, w \vDash (z_1t_1 + \cdots + z_nt_n) \geq z$ iff $(z_1t'_1 + \cdots + z_nt'_n) \geq z$, where for $1 \leq k \leq n$,

$$t'_k = \begin{cases} \mathsf{Cost}(w, A), & \text{for } t_k = c_A, \\ \mathsf{Bdg}_i(w), & \text{for } t_k = b_i. \end{cases}$$

We refer to the class of all models satisfying all properties mentioned above as $\mathfrak{M}$. We write $\vDash_{\mathfrak{M}} \varphi$ if the formula φ is *valid* in the class of models $\mathfrak{M}$.

2.3 Soundness and Completeness

The axiomatisation of the logic $\mathsf{EL}_{\mathsf{bc}}$ is presented in Table 1. Here, (Ineq) is the set of axioms for linear inequalities firstly described in [11] and used later for similar purposes in [17].

Table 1. Proof system for $\mathsf{EL}_{\mathsf{bc}}$

Axioms	
(Taut)	All instances of propositional tautologies
(Ineq)	All instances of the axioms for linear inequalities
(K)	$K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$
(T)	$K_i\varphi \to \varphi$
(4)	$K_i\varphi \to K_iK_i\varphi$
(5)	$\neg K_i\varphi \to K_i\neg K_i\varphi$
(Bd)	$b_i \geq 0$
($\geq_1$)	$c_A \geq 0$
($\geq_2$)	$c_\top = 0$
($\geq_3$)	$c_A = c_B$ if $A \approx B$, for all formulas $A, B \in \mathcal{L}_{PL}$
Inference rules	
(MP)	From φ and $\varphi \to \psi$, infer ψ
(Nec$_i$)	From φ infer $K_i\varphi$

Axioms (Ineq) allow us to prove all valid formulas about linear inequalities. These axioms are presented in Table 2.

Table 2. Axioms for reasoning about linear inequalities

(I1)	$(a_1t_1+\cdots+a_kt_k \geq c) \leftrightarrow (a_1t_1+\cdots+a_kt_k+0t_{k+1}) \geq c)$
(I2)	$(a_1t_1 + \cdots + a_kt_k \geq c) \to (a_{j_1}t_{j_1} + \cdots + a_{j_k}t_{j_k} \geq c)$, where $j_1, \ldots, j_k$ is a permutation of $1, \ldots, k$
(I3)	$(a_1t_1 + \cdots + a_kt_k \geq c) \wedge (a'_1t_1 + \cdots + a'_kt_k \geq c') \to$ $\to (a_1 + a'_1)t_1 + \cdots + (a_k + a'_k)t_k \geq (c + c')$
(I4)	$(a_1t_1 + \cdots + a_kt_k \geq c) \leftrightarrow (da_1t_1 + \cdots + da_kt_k \geq dc)$ for $d > 0$
(I5)	$(t \geq c) \vee (t \leq c)$
(I6)	$(t \geq c) \to (t > d)$, where $c > d$

Theorem 1 (Soundness). $\mathsf{EL}_{\mathsf{bc}}$ *is sound w.r.t.* $\mathfrak{M}$, *i.e.,* $\vdash_{\mathsf{EL}_{\mathsf{bc}}} \varphi \;\Rightarrow\; \models_{\mathfrak{M}} \varphi$.

Proof. Straightforward.

For the completeness proof, fix an $\mathsf{EL_{bc}}$-consistent formula φ. We start with the set $\Gamma = \mathsf{Sub}(\varphi)$ of all subformulas of φ. Next, let $\Gamma^+ \supseteq \Gamma$ be the smallest set of formulas such that

1. Γ^+ is closed under single negation: if $\psi \in \Gamma^+$ and ψ does not start with $\neg$, then $\neg\psi \in \Gamma^+$,
2. $(b_i \geq 0) \in \Gamma^+$, for each agent $i \in \mathsf{Agt}$ that occurs in Γ (in b_i or K_i),
3. $(c_A \geq 0) \in \Gamma^+$, for each constant c_A that occurs in Γ,
4. $(c_\top = 0) \in \Gamma^+$,
5. $c_A = c_B \in \Gamma^+$ for all constants c_A and c_B occurring in Γ such that $A \approx B$.

First, we build a finite *canonical pre-model* $\mathcal{M}^c = (W^c, (\sim_i^c)_{i \in \mathsf{Agt}}, V^c)$ by the construction similar to that used for the multi-agent logic **S5**:

– W^c is the set of all maximal $\mathsf{EL_{bc}}$-consistent subsets of Γ^+;
– $x \sim_i^c y$ iff, for all formulas $\psi \in \Gamma^+$, we have $K_i\psi \in x$ iff $K_i\psi \in y$;
– $w \in V^c(p)$ iff $p \in w$, for each propositional variable $p \in \Gamma$.

So far, $\mathcal{M}^c$ is a Kripke model, without the Cost^c and Bdg^c functions. Thus it remains to prove that both functions Cost^c and Bdg^c can be defined.

Since every state $w \in W^c$ is $\mathsf{EL_{bc}}$-consistent, the set of all linear inequalities contained in w is satisfiable, i.e., has at least one solution. Then we can easily construct functions $\mathsf{Cost}^c(A, w)$ and $\mathsf{Bdg}_i^c(w)$ that agree with this solution: for formulas $A \in \mathcal{L}_{PL}$ such that c_A occurs in Γ^+, we put $\mathsf{Cost}^c(A, w)$ to be the real that corresponds to c_A in that solution; for other formulas $B \in \mathcal{L}_{PL}$, if $B \approx A$ for some formula A such that c_A is in Γ, then we put $\mathsf{Cost}^c(B, w) := \mathsf{Cost}^c(A, w)$. Thus we can enforce that for all $w \in W^c$ and all $A \in \mathcal{L}_{PL}$ such that c_A occurs in Γ^+ it holds that

(1) $\mathsf{Cost}^c(A, w) \geq 0$ for all formulas $A \in \mathcal{L}_{PL}$ such that c_A occurs in Γ^+, by the construction of Γ^+ and $(\geq_1)$ axiom,
(2) $\mathsf{Cost}^c(\top, w) = 0$, by the construction of Γ^+ and $(\geq_2)$ axiom,
(3) $\mathsf{Cost}^c(A, w) = \mathsf{Cost}^c(B, w)$ for all $A, B \in \mathcal{L}_{PL}$ such that $A \approx B$, by $(\geq_3)$ axiom.

Similarly, we construct Bdg^c function such that for each $w \in W^c$ and each $i \in Agt$, $\mathsf{Bdg}_i^c(w)$ agrees with existing solution of linear inequalities, contained in w. This construction is well-defined and for any $w \in W^c$ and any $i \in \mathsf{Agt}$, it holds that

(1) $\mathsf{Bdg}_i^c(w) \geq 0$ by axiom (Bd) and the construction of Γ^+,
(2) $\mathsf{Bdg}_i^c(w) \geq \mathsf{Cost}^*(A)$ iff $(b_i \geq c_A) \in w$, for all b_i, c_A in Γ.

Thus, we obtained a finite canonical model $\mathcal{M}^c = (W^c, (\sim_i^c)_{i \in \mathsf{Agt}}, \mathsf{Cost}^c, \mathsf{Bdg}^c, V^c)$. As we have already demonstrated, this model satisfies the properties (C1) and (C2). It is also clear that for all $i \in \mathsf{Agt}$, $\sim_i^c$ is an equivalence relation on W^c.

Lemma 1 (Truth Lemma). *For any $\psi \in \Gamma^+$, we have: $\mathcal{M}^c, w \vDash \psi \iff \psi \in w$.*

Proof. Induction on ψ. Cases for $p \in \mathsf{Prop}$ and Boolean connectives are trivial.

Case $K_i\psi$:
$\mathcal{M}^c, w \vDash K_i\psi$ iff $\forall w' : w \sim_i^c w' \Rightarrow M^c, w' \vDash \psi$ by Definition 3. $\forall w' : w \sim_i^c w' \Rightarrow$

$M^c, w' \vDash \psi$ iff $\forall w' : w \sim_i^c w' \Rightarrow \psi \in w'$ by previous induction step. $\forall w' : w \sim_i^c w' \Rightarrow \psi \in w'$ iff $K_i\psi \in w$ by the construction of $\sim_i^c$.

Case $(z_1 t_1 + \cdots + z_n t_n) \geq z$:
$\mathcal{M}^c, w \vDash (z_1 t_1 + \cdots + z_n t_n) \geq z$ iff $(z_1 t'_1 + \cdots + z_n t'_n) \geq z$, where $t'_1, \ldots, t'_n$ are represented by $\mathsf{Cost}^c(A)$ and $\mathsf{Bdg}_i^c(w)$ for the corresponding constants c_A and b_i that occur in $(z_1 t_1 + \cdots + z_n t_n) \geq z$. By the construction of Cost^c and Bdg^c, it also holds that $(z_1 t'_1 + \cdots + z_n t'_n) \geq z$ iff $(z_1 t_1 + \cdots + z_n t_n) \geq z \in w$.

Theorem 2 (Completeness). $\mathsf{EL_{bc}}$ *is complete w.r.t.* $\mathfrak{M}$*, i.e.,* $\vDash_{\mathfrak{M}} \varphi$ *iff* $\vdash_{\mathsf{EL_{bc}}} \varphi$.

Proof. The right-to-left direction follows from Theorem 1. For the left-to-right direction, consider a formula φ such that $\nvdash_{\mathsf{EL_{bc}}} \varphi$. Construct a model $\mathcal{M}^c$ for $\neg\varphi$. From Lemma 1 it is clear that $\exists w \in W^c$ such that $\mathcal{M}^c, w \vDash \neg\varphi$. Then $\mathcal{M}^c, w \nvDash \varphi$. It is also clear that $\mathcal{M}^c \in \mathfrak{M}$, by the construction of $\mathcal{M}^c$, so $\nvDash_{\mathfrak{M}} \varphi$.

Here we should also mention that in $\mathsf{EL_{bc}}$ we intentionally impose as less semantic restrictions as possible to deal with the most general case. In particular, we assume that it is possible that an agent does not know her own budget. But this restriction can be imposed by adding the following axiom to $\mathsf{EL_{bc}}$:

$$(b_i = z) \rightarrow K_i(b_i = z) \tag{Kb}$$

Let $\mathfrak{M}^{\mathrm{Kb}}$ be a subclass of $\mathfrak{M}$ such that for any $w_1, w_2 \in W : w_1 \sim_i w_2 \Rightarrow Bdg_i(w_1) = Bdg_i(w_2)$. Then it is straightworfard to prove the following result.

Theorem 3 (Completeness). *The logic* $\mathsf{EL_{bc}} + \mathrm{Kb}$ *is complete with respect to* $\mathfrak{M}^{\mathrm{Kb}}$*, i.e.,* $\vDash_{\mathfrak{M}^{\mathrm{Kb}}} \varphi \quad\Leftrightarrow\quad \vdash_{\mathsf{EL_{bc}}+\mathrm{Kb}} \varphi$.

Note also that here we prove only weak completeness result due to non-compactness of $\mathsf{EL_{bc}}$. To see that $\mathsf{EL_{bc}}$ is non-compact consider a set of $\mathsf{EL_{bc}}$-formulas: $\{c_A > n \mid n \in \mathbb{N}\}$. It is easy to see that any finite subset of this set is satisfiable while the set itself is not.

Theorem 4 (Decidability). *The satisfiability problem for* $\mathsf{EL_{bc}}$ *is decidable.*

Proof. In this proof we follow the technique similar to those from [7]. From the proof of Theorem 2 it follows that a formula φ is satisfiable iff it is satisfiable in a model $\mathcal{M} \in \mathfrak{M}$ with at most $2^{|\Gamma^+|}$ states. However, since these models include Cost and Bdg functions there are infinitely many of them. In order to restrict the set of structures to check to be finite, we will consider pseudo-models which do not have Cost and Bdg, but it is easy to check whether a corresponding functions exist. We call pseudo-models for which both Cost and Bdg exist solvable. The existence of one of such solvable pseudo-models satisfying φ will guarantee the existence of a proper model (for which Cost and Bdg are defined) that satisfies φ.

Consider a set Γ^+ defined in the proof of Theorem 2 and let a set $Sum(\varphi)$ be a set of all elements of Γ^+ of the form $\sum_{k=1}^{n} z_k t_k \geq z$. For every $l \leq 2^{|\Gamma^+|}$ we consider a pseudo-model $\overline{\mathcal{M}} = (\overline{W}, \overline{\sim_i}, \overline{S}, \overline{V})$, where $\overline{W}, \overline{\sim_i}$ and $\overline{V}$ are defined in a standard way and $\overline{S}$ is defined as follows:

$$\overline{S} : \overline{W} \times Sum(\varphi) \longrightarrow \{true, false\}.$$

Note that there are only finitely many pseudo-models for each l. They are not models of our logic, but we can check if an element of Γ^+ holds in some states of this pseudo-model using the $\vDash'$ relation which is defined in a trivial way, except the case for $\sum\limits_{k=1}^{n} z_k t_k \geq z$:

$$\overline{\mathcal{M}}, w \vDash \sum_{k=1}^{n} z_k t_k \geq z \text{ iff } \overline{S}(w, \sum_{k=1}^{n} z_k t_k \geq z) = true.$$

We will consider only those pseudo-models $\overline{\mathcal{M}}$ such that $\overline{\mathcal{M}}, w \vDash' \varphi$ for some $w \in \overline{W}$. For each such $\overline{\mathcal{M}}$ we want to check whether $\overline{\mathcal{M}}$ can be extended to a structure $\mathcal{M} \in \mathfrak{M}$ of our logic. In other words, we want to check if $\overline{S}$ can be replaced by a tuple $(\mathsf{Cost}, \mathsf{Bdg})$ that agrees with $\overline{S}$ and for every $w \in W$ and every $\psi \in Sum(\varphi)$ we have $\mathcal{M}, w \vDash \psi$ iff $\overline{S}(w, \psi) = true$. It is straightforward to check that for such $\mathcal{M}$ it holds that $\mathcal{M}, w \vDash \chi$ iff $\overline{\mathcal{M}}, w \vDash' \chi$ for every $\chi \in \Gamma^+(\varphi)$. For this purpose we consider special system of linear inequalities to define $\mathsf{Cost}(A, w)$ and $\mathsf{Bdg}_i(w)$ for each $i \in Agt$ and each $w \in W$. We use the variables of the form $c_{\chi,w}$ and $b_{i,w}$ which represent the values of $\mathsf{Cost}(\chi, w)$ and $\mathsf{Bdg}_i(w)$ respectively. Now we are ready to define a system of linear inequalities:

(1) $c_{\chi,w} \geq 0$ for each $\chi \in \mathcal{L}_{PL} \cap \Gamma^+$ and $w \in W$,
(2) $b_{i,w} \geq 0$ for each $i \in Agt$ and each $w \in W$,
(3) $c_{\top,w} = 0$ for each $w \in W$,
(4) $c_{\chi,w} = c_{\chi',w}$ for each $w \in W$, where $\chi, \chi' \in \mathcal{L}_{PL} \cap \Gamma^+$ such that $\chi \approx \chi'$,
(5) $\sum\limits_{k=1}^{n} z_k t_k \geq z$, where each occurrence of c_A and b_i are replaced with $c_{A,w}$ and $b_{i,w}$ for every formula $\sum\limits_{k=1}^{n} z_k t_k \geq z$ such that $\overline{S}(w, \sum\limits_{k=1}^{n} z_k t_k \geq z) = true$,
(6) $\sum\limits_{k=1}^{n} z_k t_k < z$, where each occurrence of c_A and b_i are replaced with $c_{A,w}$ and $b_{i,w}$ for every formula $\sum\limits_{k=1}^{n} z_k t_k \geq z$ such that $\overline{S}(w, \sum\limits_{k=1}^{n} z_k t_k \geq z) = false$.

For our purposes it is sufficient to find at least one solution of such system of equations and inequalities. Note that this system is finite and the problem of solving systems of inequalities is decidable. So, given a pseudo-model we can check if this pseudo-model is solvable (by solving a corresponding system of inequalities). It is straightforward to see that if there is a solvable pseudo-model for φ, then φ is satisfiable.

The proof for other direction is trivial, since the canonical model for φ gives rise to a solvable pseudo-model with $2^{|\Gamma^+|}$ states. Then if φ is satisfiable, then there is a solvable pseudo-model for φ with $l \leq 2^{|\Gamma^+|}$ states.

We have shown that φ is satisfiable iff there is a solvable pseudo-model for φ with $l \leq 2^{|\Gamma^+|}$ states. So, we can check satisfiablity of φ examining finitely many choices of l for which there are only finitely many pseudo-models and each pseudo-model can be verified to be solvable in a finite number of steps.

3 Dynamic Epistemic Logic for Budget-Constrained Agents

The dynamic language $\mathsf{DEL_{bc}}$ extends the static language $\mathsf{EL_{bc}}$ with a dynamic operator $[?_i A]\varphi$. A formula $[?_i A]\varphi$ can be read as *"φ is true after i's question whether A is true"*.

3.1 Syntax

Definition 4. The *formulas* of $\mathsf{DEL_{bc}}$ are defined by the following grammar:

$$\varphi, \psi ::= p \mid (z_1 t_1 + \cdots + z_n t_n) \geq z) \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i\varphi \mid [?_i A]\varphi,$$

where $p \in \mathsf{Prop}$, $A \in \mathcal{L}_{PL}$, $i \in Agt$, $t_1, \ldots, t_n \in \mathsf{Const}$ and $z_1, \ldots, z_n, z \in \mathbb{Z}$.

The dual operator $\langle ?_i A\rangle\varphi$ can be defined in a standard way: $\langle ?_i A\rangle\varphi \equiv \neg[?_i A]\neg\varphi$.

3.2 Semantics

The main features of the operator $[?_i A]\varphi$ are: (1) every agent knows that the question was asked, i.e., the very fact of the question is public, (2) only the agent i knows the answer, i.e., the answer is private, (3) the question requires the agent i to spend some amount of resources. All of these features will be described formally in this section.

We extend the truth relation $\vDash$ introduced in Definition 3 to the dynamic operator $[?_i A]\varphi$ as follows.

Definition 5. Given a model $\mathcal{M} = (W, (\sim_i)_{i\in\mathsf{Agt}}, \mathsf{Cost}, \mathsf{Bdg}, V)$ and a state $w \in W$,

$$\mathcal{M}, w \vDash [?_i A]\varphi \quad \text{iff} \quad \mathcal{M}, w \vDash (b_i \geq c_A) \text{ implies } \mathcal{M}^{?_i A}, w \vDash \varphi.$$

Here $\mathcal{M}^{?_i A}$ is a model obtained from $\mathcal{M}$ by the update that corresponds to the following action: "*the agent i asked whether the propositional formula A is true and spent for this the amount of resources* $\mathsf{Cost}(A)$"; the updated model is described in the next definition. We will use notation: $[A]_\mathcal{M} := \{w \in W \mid \mathcal{M}, w \vDash A\}$.

Definition 6. Given a model $\mathcal{M} = (W, (\sim_i)_{i\in\mathsf{Agt}}, \mathsf{Cost}, \mathsf{Bdg}, V)$, an *updated model* is a tuple $\mathcal{M}^{?_i A} = (W^{?_i A}, (\sim_j^{?_i A})_{j\in\mathsf{Agt}}, \mathsf{Cost}^{?_i A}, \mathsf{Bdg}^{?_i A}, V^{?_i A})$, where

- $W^{?_i A} = \{w \in W \mid \mathcal{M}, w \models b_i \geq c_A\}$,
- $\sim_j^{?_i A} = (W^{?_i A} \times W^{?_i A}) \cap \sim_j^*$,

  where $\sim_j^* = \begin{cases} \sim_j \bigcap\Big(([A]_\mathcal{M} \times [A]_\mathcal{M}) \bigcup([\neg A]_\mathcal{M} \times [\neg A]_\mathcal{M})\Big) & \text{if } j = i, \\ \sim_j & \text{otherwise,} \end{cases}$

- $\mathsf{Cost}^{?_i A}(B) = \mathsf{Cost}(B)$, for all propositional formulas B,
- $\mathsf{Bdg}_j^{?_i A}(w) = \begin{cases} \mathsf{Bdg}_j(w) - \mathsf{Cost}(A, w), & \text{if } j = i, \\ \mathsf{Bdg}_j(w), & \text{otherwise,} \end{cases}$
- $V^{?_i A}(p) = V(p) \cap W^{?_i A}$.

Intuitively, the update $[?_i A]\varphi$ of model $\mathcal{M}$ firstly removes all states of $\mathcal{M}$ in which agent i does not have a sufficient amount of resources to ask about A. This can be justified by the fact that other agents do not necessarily know i's budget, but when they observe the fact that i actually asks about the truth of A, it no longer makes sense to consider the states with $(b_i < c_A)$ as possible ones. Secondly, when i asks *"is A true?"*, she gets either *"Yes"* or *"No"* and we consider this fact to be known by all agents. Then, after this update, the agent i necessarily distinguishes any two states of $\mathcal{M}$ that do not agree on the valuation of A. But since the actual answer is available only to the agent i, the epistemic relations of other agents remain the same, only taking into account that some states have been removed. This update does not affect the costs of formulas and budgets of all agents except i. Budget of i decreases by the cost of A after $[?_i A]$. As one can see, all of these assumptions sound quite natural.

Consider an example with two agents i and j. Let p_i stand for'i is COVID-positive' and p_j stands for'j is COVID-positive'. Assume that the cost of the test is 20 resources in all possible worlds ($\mathcal{M} \vDash c_{p_i} = 20 \wedge c_{p_j} = 20$). If we also assume that i decides to make the test ($[?_i p_i]$), then the semantics of $\mathsf{DEL_{bc}}$ describes this situation as presented in Fig. 1.

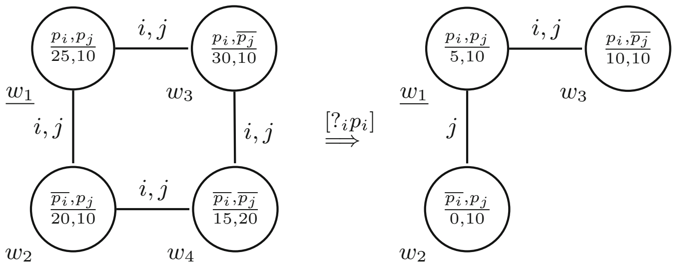

Fig. 1. Initial model $\mathcal{M}$ and updated model $\mathcal{M}^{?_i p_i}$

Note that an agent does not necessarily knows even her own budget. The following formulas hold in w_1:

- $\mathcal{M}, w_1 \vDash \neg K_i p_i$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_i p_i$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash \neg K_j p_i$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_j K_i^? p_i$
- $\mathcal{M}, w_1 \vDash \neg K_i (b_i \geq 20)$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_i (b_i \geq 0)$
- $\mathcal{M}, w_1 \vDash \neg K_j (b_j = 10)$
- $\mathcal{M}^{?_i p_i}, w_1 \vDash K_j (b_j = 10)$.

3.3 Some Valid Formulas

Here we present some examples of valid formulas w.r.t. the proposed semantics.

Proposition 1. $\vDash (b_i \geq c_A) \leftrightarrow \langle ?_i A\rangle\top$.

Proof. $\mathcal{M}, w \vDash \langle ?_i A\rangle\top$ is equivalent to $\mathcal{M}, w \vDash \neg[?_i A]\bot$ by definition of $\langle ?_i A\rangle$. Then $\mathcal{M}, w \vDash \neg[?_i A]\bot$ is equivalent to $\mathcal{M}, w \vDash (b_i \geq c_A)$ and $\mathcal{M}^{?_i A}, w \vDash \top$. But since $w \in W^{?_i A}$ iff $\mathcal{M}, w \vDash (b_i \geq c_A)$, then $\mathcal{M}^{?_i A}, w \vDash \top$ is also equivalent to $\mathcal{M}, w \vDash (b_i \geq c_A)$.

Proposition 2. $\vDash \langle ?_i A\rangle\varphi \rightarrow [?_i A]\varphi$.

Proof. As we mentioned above, $\mathcal{M}, w \vDash \langle ?_i A\rangle\varphi$ is equivalent to $\mathcal{M}, w \vDash (b_i \geq c_A)$ and $\mathcal{M}^{?_i A}, w \vDash \varphi$. This conjunction obviously implies that $\mathcal{M}, w \vDash (b_i \geq c_A) \Rightarrow \mathcal{M}^{?_i A}, w \vDash \varphi$.

Proposition 3. $\vDash [?_i A]K_i^? A$.

Proof. It is clear that $w \sim^{?_i A} w'$ implies ($\mathcal{M}, w \vDash A$ and $\mathcal{M}, w' \vDash A$) or ($\mathcal{M}, w \vDash \neg A$ and $\mathcal{M}, w' \vDash \neg A$) by Definition 6. Then $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_i A}, w \vDash (K_i A \vee K_i \neg A)$.

3.4 Soundness and Completeness

Axiomatization of $\mathsf{DEL_{bc}}$ can be obtained by adding the reduction axioms from Table 3 to the axiomatization of $\mathsf{EL_{bc}}$. The notation $\big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big)^{[b_i \backslash (b_i - c_A)]}$ means that all occurrences of b_i in $(z_1 t_1 + \cdots + z_n t_n) \geq z$ are replaced with $(b_i - c_A)$.

Table 3. Reduction axioms and inference rules

(R_p)	$[?_i A]p \leftrightarrow (b_i \geq c_A) \rightarrow p$
$(R_\geq)$	$[?_i A]\big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big) \leftrightarrow (b_i \geq c_A) \rightarrow$ $\rightarrow \big((z_1 t_1 + \cdots + z_n t_n) \geq z)\big)^{[b_i \backslash (b_i - c_A)]}$
$(R_\neg)$	$[?_i A]\neg\varphi \leftrightarrow (b_i \geq c_A) \rightarrow \neg[?_i A]\varphi$
$(R_\wedge)$	$[?_i A](\varphi \wedge \psi) \leftrightarrow [?_i A]\varphi \wedge [?_i A]\psi$
(R_{K_j})	$[?_i A]K_j\varphi \leftrightarrow (b_i \geq c_A) \rightarrow K_j[?_i A]\varphi$, where $i \neq j$
(R_{K_i})	$[?_i A]K_i\varphi \leftrightarrow (b_i \geq c_A) \rightarrow$ $\rightarrow \Big(\big(A \rightarrow K_i(A \rightarrow [?_i A]\varphi)\big) \wedge \big(\neg A \rightarrow K_i(\neg A \rightarrow$ $[?_i A]\varphi)\big)\Big)$
(Rep)	From $\vdash \varphi \leftrightarrow \psi$, infer $\vdash [?_i A]\varphi \leftrightarrow [?_i A]\psi$

Proposition 4. *Axioms* (R_p), $(R_\neg)$, *and* $(R_\wedge)$ *and inference rule Rep are sound w.r.t.* $\mathfrak{M}$.

Proof. Trivial.

Lemma 2. *For $i \neq j$ is holds that $w \sim_j^{?_iA} w'$ iff $w \sim_j w'$, $\mathcal{M}, w \vDash (b_i \geq c_A)$, and $\mathcal{M}, w \vDash (b_i \geq c_A)$.*

Proof. Follows straightforward from Definition 6.

Proposition 5. *For any model $\mathcal{M}$ and any point $w \in W$, it holds that*

$$\mathcal{M}, w \vDash [?_iA]K_j\varphi \text{ iff } \mathcal{M}, w \vDash (b_i \geq c_A) \to K_j[?_iA]\varphi, \text{ where } i \neq j.$$

Proof ($\Rightarrow$). Let $\mathcal{M}, w \vDash [?_iA]K_j\varphi$ (1) and $\mathcal{M}, w \vDash (b_i \geq c_A)$ (2). From (1), $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_iA}, w \vDash K_j\varphi$ (1.1) by Definition 5. Then $\mathcal{M}^{?_iA}, w \vDash K_j\varphi$ from (1.1) and (2). Then $\forall w' : (w \sim_j^{?_iA} w') \Rightarrow \mathcal{M}^{?_iA}, w' \vDash \varphi$ by Definition 3. This fact together with Lemma 2 implies that $\forall w'(w \sim_j w')$: $\mathcal{M}, w' \vDash (b_i \geq c_A) \Rightarrow \mathcal{M}^{?_iA}, w' \vDash \varphi$. This is equivalent to $\mathcal{M}, w \vDash K_j[?_iA]\varphi$, by Definition 3 and Definition 5.

($\Leftarrow$). The case for $\mathcal{M}, w \nvDash (b_i \geq c_A)$ is trivial. Consider only the case for $\mathcal{M}, w \vDash K_j[?_iA]\varphi$. Then $\forall w'(w \sim_j w') : \mathcal{M}, w' \vDash (b_i \geq c_A) \Rightarrow \mathcal{M}^{?_iA}, w' \vDash \varphi$. By Lemma 2 it holds that $\forall w' : w \sim_j^{?_iA} w' \Rightarrow \mathcal{M}^{?_iA}, w' \vDash \varphi$. By Definition 3, $\mathcal{M}^{?_iA}, w \vDash K_j\varphi$ and hence $\mathcal{M}, w \vDash [?_iA]K_j\varphi$.

Lemma 3.

– *$w \sim_i^{?_iA} w'$ iff $w \sim_i w'$ (1), $\mathcal{M}, w \vDash (b_i \geq c_A)$ (2.1), $\mathcal{M}, w' \vDash (b_i \geq c_A)$ (2.2) and $w \approx_A w'$ (3), where $w \approx_A w'$ holds if either both $\mathcal{M}, w \vDash A$ and $\mathcal{M}, w' \vDash A$ hold or both $\mathcal{M}, w \vDash \neg A$ and $\mathcal{M}, w' \vDash \neg A$ hold,*
– *$\mathcal{M}, w \vDash A$ iff $\mathcal{M}^{?_iA}, w \vDash A$, where A is a propositional formula.*

Proof. Follows straightforwardly from Definition 6.

Proposition 6. *For any model $\mathcal{M}$ and any point $w \in W$, we have:*

$$\mathcal{M}, w \vDash [?_iA]K_i\varphi \text{ iff } \mathcal{M}, w \vDash (b_i \geq c_A) \to \bigwedge_{A' \in \{A, \neg A\}} \big(A' \to K_i(A' \to [?_iA']\varphi)\big).$$

Proof ($\Rightarrow$). Let $\mathcal{M}, w \vDash [?_iA]K_i\varphi$ (1) and $\mathcal{M}, w \vDash (b_i \geq c_A)$ (2). From (1), (2) and Definition 5 we get $\mathcal{M}^{?_iA}, w \vDash K_i\varphi$. Then $\forall w'(w \sim_i^{?_iA} w') \Rightarrow \mathcal{M}^{?_iA}, w' \vDash \varphi$. Assume that $\mathcal{M}, w \vDash A$. Then by Lemma 3 it follows that $\forall w' : w \sim_i w'$ and $\mathcal{M}, w' \vDash A$ and $\mathcal{M}, w' \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_iA}, w' \vDash \varphi$. This is equivalent to $\mathcal{M}, w \vDash K_i(A \to [?_iA]\varphi)$ by Definition 3 and Definition 5. Then, from our assumption we proved that $\mathcal{M}, w \vDash A \to K_i(A \to [?_iA]\varphi)$. By a similar argument, one can show that $\mathcal{M}, w \vDash \neg A \to K_i(\neg A \to [?_iA]\varphi)$.

($\Leftarrow$). The case for $\mathcal{M}, w \nvDash (b_i \geq c_A)$ is trivial. Consider only the case for $\mathcal{M}, w \vDash \bigwedge_{A' \in \{A, \neg A\}} \big(A' \to K_i(A' \to [?_iA']\varphi)\big)$. Assume that $\mathcal{M}, w \vDash A$. Then $\mathcal{M}, w \vDash K_i(A \to [?_iA]\varphi)$. Similarly, assuming $\mathcal{M}, w \vDash \neg A$ entails $\mathcal{M}, w \vDash K_i(\neg A \to [?_iA]\varphi)$. Then for all w', such that $(w \sim_i w')$ and w' agrees with w on the valuation of A it holds that $\mathcal{M}, w' \vDash [?_iA]\varphi$ and hence $\mathcal{M}, w' \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_iA}, w' \vDash \varphi$. Then by Lemma 3 it holds that $\forall w' : w \sim_i^{?_iA} w' \Rightarrow \mathcal{M}^{?_iA}, w' \vDash \varphi$. And hence $\mathcal{M}^{?_iA}, w \vDash K_i\varphi$. By Definition 5, the last claim implies $\mathcal{M}, w \vDash [?_iA]K_i\varphi$.

Proposition 7. *Axiom* $(R_{\geq})$ *is sound w.r.t.* $\mathfrak{M}$

Proof. It is clear that $\mathcal{M}, w \vDash [?_i A](z_1 t_1 + \cdots + z_n t_n) \geq z$ iff $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}^{?_i A}, w \vDash (z_1 t_1 + \cdots + z_n t_n) \geq z$ by Definition 5. Note that $\mathcal{M}^{?_i A}, w \vDash (z_1 t_1 + \cdots + z_n t_n) \geq z$ is equivalent to $\mathcal{M}, w \vDash (z_1 t_1^* + \cdots + z_n t_n^*) \geq z$, where $t_k^* = t_k$ for $t_k = c_A$ or $t_k = b_j$. And $t_k^* = t_k + \mathsf{Cost}(A)$ for $t_k = b_i$ since $\mathsf{Cost}^{?_i A}(B) = \mathsf{Cost}(B)$, $\mathsf{Bdg}_j^{?_i A}(w) = \mathsf{Bdg}_j(w)$ for $i \neq j$ and $\mathsf{Bdg}_i^{?_i A}(w) = \mathsf{Bdg}_i(w) - \mathsf{Cost}(A)$. Then $\mathcal{M}, w \vDash [?_i A](z_1 t_1 + \cdots + z_n t_n) \geq z$ iff $\mathcal{M}, w \vDash (b_i \geq c_A)$ implies $\mathcal{M}, w \vDash [(z_1 t_1 + \cdots + z_n t_n) \geq z)]^{[b_i \backslash (b_i - c_A)]}$.

Theorem 5 (Soundness). $\mathsf{DEL_{bc}}$ *is sound w.r.t.* $\mathfrak{M}$, *i.e.* $\vdash_{\mathsf{DEL_{bc}}} \varphi \Longrightarrow \vDash_{\mathfrak{M}} \varphi$.

Proof. Follows from Proposition 4–Proposition 7.

Theorem 6 (Completeness). $\mathsf{DEL_{bc}}$ *is complete w.r.t.* $\mathfrak{M}$, *i.e.* $\vdash_{\mathsf{DEL_{bc}}} \varphi$ *iff* $\vDash_{\mathfrak{M}} \varphi$.

Proof. Left-to-right direction follows from Theorem 5. The other direction holds by Theorem 2 and the standard for dynamic epistemic logic completeness via reduction argument.

Theorem 7 (Decidability). *The satisfiability problem for* $\mathsf{DEL_{bc}}$ *is decidable.*

Proof. This result is straightforward since any $\mathsf{DEL_{bc}}$ formula can be translated into $\mathsf{EL_{bc}}$ formula in finitely many steps by the rules presented in Table 3 and the decidability of $\mathsf{EL_{bc}}$ is demonstrated in Theorem 4.

4 Combination of $\mathsf{DEL_{bc}}$ and PAL

The language $\mathsf{DEL_{bc!}}$ extends the language $\mathsf{DEL_{bc}}$ with a standard operator for public announcement $[!\varphi]$. A formula $[!\varphi]\psi$ stands for "after public announcement of φ, it holds that ψ".

Definition 7. The *formulas* of $\mathsf{DEL_{bc!}}$ are defined by the following grammar:

$$\varphi, \psi ::= p \mid (z_1 t_1 + \cdots + z_n t_n) \geq z) \mid \neg\varphi \mid (\varphi \wedge \psi) \mid K_i \varphi \mid [?_i A]\varphi, \mid [!\varphi]\psi$$

where $p \in \mathsf{Prop}$, $A \in \mathcal{L}_{PL}$, $i \in Agt$, $t_1, \ldots, t_n \in \mathsf{Const}$ and $z_1, \ldots, z_n, z \in \mathbb{Z}$.

Definition 8. $\mathcal{M}, w \models [!\varphi]\psi \iff \mathcal{M}, w \models \varphi \Rightarrow \mathcal{M}^{!\varphi}, w \models \psi$, where $\mathcal{M}$ is defined in Definition 2 and $\mathcal{M}^{!\varphi}$ is a model $\mathcal{M}$ restricted to φ-worlds.

Rational Question. We will call the question rational if the agent doesn't know the answer to this question. We can express the condition for a rational question in $\mathsf{DEL_{bc!}}$ as $[?_i^r A]\varphi := [!\neg K_i^? A][?_i A]\varphi$. A formula $[?_i^r A]\varphi$ can be read as "φ is true after i's rational question whether A is true".

Example [3 cards puzzle]**.** From a pack of three known cards X, Y, Z, Alice, Bob and Cath each draw one card. Initially, all agents has zero points. If an agent has X or Y, then its score increases by one point. Also, from a pack of three known card $1, 0, 0$ each agent draws one card. If an agent has 1, then its score increases by one point, 0 does not change anything. An agent may ask a question publicly and get an answer (yes or no) privately. The cost of any question is 1 point. Bob asks: "Whether Cath has the card Y?". Alice says "I know that my points and Bob's points are different". Cath says "I know the cards".

We can represent the initial situation with a Fig. 2. The sequence of updates can be formalized as follows:

$$\langle ?^r_b Y_c\rangle\langle !K_a(b_a \neq b_b)\rangle\langle !K_c(XYZ)_?\rangle\top$$

Here $K_i(XYZ)_? := K^?_i X_? \wedge K^?_i Y_? \wedge K^?_i Z_?$ and $K^?_i X_? := K^?_i X_a \wedge K^?_i X_b \wedge K^?_i X_c$ (similarly for Y and Z). The results of updates are presented in Fig. 3. Hence, the only one possible world satisfies this series of updates.

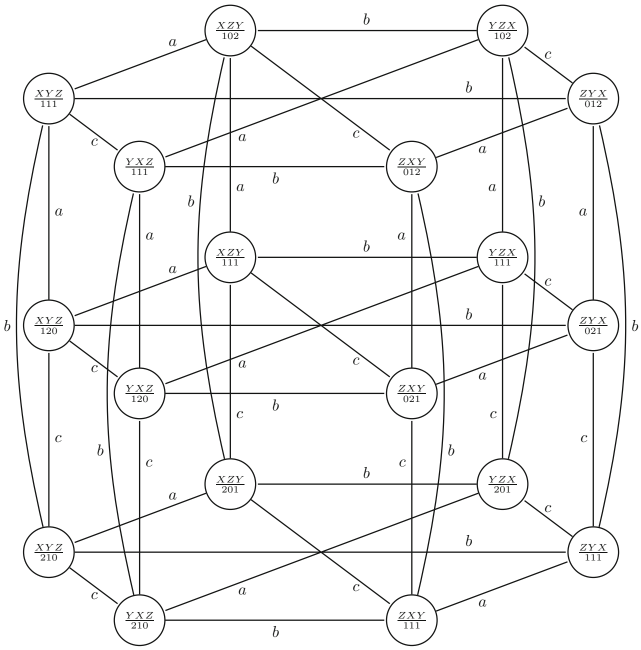

Fig. 2. Model for "3 cards" puzzle. Reflexivity, symmetry and transitivity are assumed.

Axiomatisation. The sound and complete axiomatisation for $\mathsf{DEL}_{\mathsf{bc!}}$ can be obtained as a combination of $\mathsf{DEL}_{\mathsf{bc}}$ and PAL (see [18]) proof systems with an additional reduction axiom: $[!\varphi]((z_1t_1 + \cdots + z_nt_n) \geq z) \leftrightarrow (\varphi \rightarrow (z_1t_1 + \cdots + z_nt_n) \geq z)$.

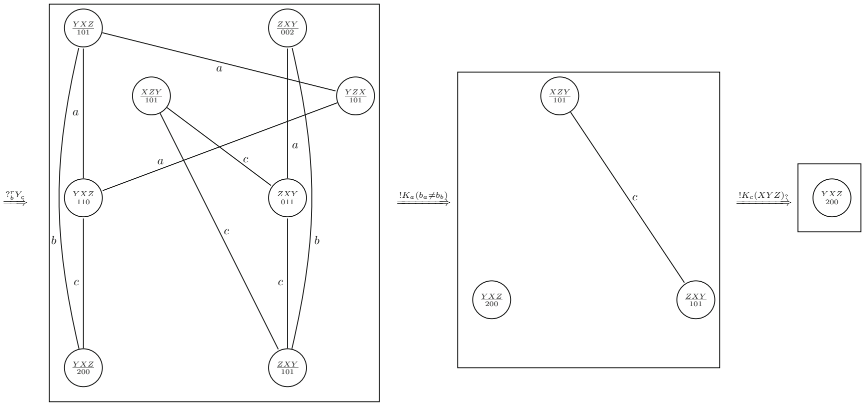

Fig. 3. Models for "3 cards" puzzle in a series of updates. Reflexivity, symmetry and transitivity are assumed.

5 Discussion

In this paper, we present $\mathsf{EL_{bc}}$, a static epistemic logic for budget-constrained agents, and provide its sound and complete axiomatisation. Then we present $\mathsf{DEL_{bc}}$, a dynamic epistemic logic for budget-constrained agents, which extends $\mathsf{EL_{bc}}$ with dynamic operator $[?_i A]\varphi$. For the dynamic fragment, we provide sound reduction axioms demonstrating $\mathsf{DEL_{bc}}$ completeness via a reduction argument. The proposed logics are sufficiently expressive to deal with non-trivial epistemic scenarios involving reasoning about costs of propositional formulas and agents' budgets. In addition, $\mathsf{DEL_{bc}}$ is able to describe the semantics of a special class of questions. These questions can be asked publicly, but the answer is available only to the asking agent. Moreover, to get an answer, an agent must spend some resources, thus decreasing her budget. This gives rise to a new direction of research in the field of reasoning about resource-bounded agents in multi-agent systems allowing to formalise not only inner or cognitive resources, but also external resources as obstacles in the process of obtaining new information from the environment.

It is worth noting that we make some assumptions about the properties of Cost and Bdg functions. Firstly, we assume that costs of formulas depend on a particular state of a model, i.e. some formula can have different costs in different states. This assumption allows us to model situations in which an agent does not necessarily know the cost of some formula. Our second assumption is that agents do not necessarily know the budget of other agents as well as their own. But this assumption can be eased by introducing additional axioms as we demonstrate in Theorem 6. Our last assumption deals with the relationship between the costs of different formulas. The fact that equivalent formulas must have equal costs seems obvious. It is also plausible that $\mathsf{Cost}(A)$ must be equal to $\mathsf{Cost}(\neg A)$, since asking questions "Is A true?" and "Is $\neg A$ true?" can be considered as the same informational action. But these are the only constraints on the Cost function we imposed in this paper. It remains an open question how to deal with Boolean connectives in the sense of their costs. As a future work, one of our aims is to deal with this aspect. For example, it looks quite natural to consider the following property: $c_A + c_B \geq c_{A \circ B}$, where $\circ$ is any Boolean connective.

As for the $\mathsf{DEL_{bc}}$ extension, it is natural to introduce additional dynamic modalities: an operator $[?_G A]\psi$ which involves sharing resources among a group of agents, G and an operator $\langle ?_i^n \rangle \varphi$ for existential quantification over updates (there is a propositional formula, A, such that the cost of A is at most, n, and it is true that $\langle ?_i A \rangle \varphi$). This would allow us to define a concept such as n-knowability, meaning that φ is knowable given n resources. Finally, in future work, we plan to establish complexity results for the satisfiability problem and investigate model-checking algorithms for our logics.

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